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Calculations for anemometry with fine hot wires

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The heat transfer appropriate to low Reynolds number hot-wire anemometry is calculated from the full non-linear equations of motion and of heat transfer by an iterative method starting with the Oseen solution and its heat flux analogue. The second and third iterates yield close agreement with measured data.

1. Introduction

When a hot-wire anemometer is calibrated with the wire effectively normal to a steady uniform stream and is used in an oblique or unsteady flow, it is desirable to know how accurately the calibration heat loss can be ascribed to two-dimensional forced convection. End effects, if present, may impair the accuracy of the cosine rule for oblique velocities (Corrsin 1963) or the response time in unsteady flow. The cosine rule is also invalid if buoyancy is significant. To expose these and other extraneous sources of heat loss, it is useful to have at hand comparative heat transfer data, that is, data on forced, two-dimensional convection from circular cylinders with uniformly hot surfaces.

For sufficiently small Reynolds numbers R the relatively few available experimental measurements can be supplemented fairly readily by calculation. The convection velocity in the heat equation may be replaced, to a first approximation, by the free-stream velocity (Cole & Roshko 1954), as in Oseen's equation for the viscous flow. The resulting Nusselt numbers, one variant of which denoted N_1 appears below, are of $O(\log^{-1} R)$ with an error generally of $O(\log^{-3} R)$. Higher approximations follow after correcting iteratively for the convection velocity in the temperature equation, and, when needed, in the vorticity equation. The present paper adds the first two of these succeeding Nusselt number approximations, N_2 and N_3, which are in error by $O(\log^{-4} R)$ and $O(\log^{-5} R)$ respectively.

The N_1, N_2 and N_3 evaluated for a Prandtl number σ of 0·72 and diameter-based Reynolds numbers R up to 0·8 are shown in figure 1, together with the measured data of Collis & Williams (1959). End effects were kept small in their experiments and corrections were made both for metallic conduction along the wire and for temperature jump at its surface. Natural convection could be recognized when results for different wire diameters were correlated and those results perceptibly affected by buoyancy were discounted. It will be seen that for Reynolds numbers R less than 0·3, the second and third approximations N_2 and N_3 lie within 1 % of the experimental values. For R beyond 0·3, the calculated and experimental values diverge rapidly.

Because of the susceptibility of the heat loss at small Reynolds numbers to buoyancy, it may be worth recapitulating Collis & Williams's (1959) criterion that for horizontal wires of large aspect ratio the effects of free convection can be ignored when R exceeds $1{\cdot}09G^{0{\cdot}39}(1+T_w/T_\infty)^{0{\cdot}76}$, where T_w and T_∞ denote respectively the wire and the ambient temperature and G denotes the Grashof number based on diameter and ambient values of the physical constants. This criterion was derived for Reynolds numbers less than about 0·1. For a discussion of the relevant experimental evidence the reader is referred to the original paper.

2. Velocity approximation

We shall consider the steady convection of heat from a circular cylinder of uniform surface temperature T_w in an air current of temperature T_∞ and speed U_∞. The density ρ, viscosity ν, conductivity k and specific heat c_p of the air will be regarded as constants, and viscous heating will be ignored. As units of speed and length we adopt respectively U_∞ and the cylinder diameter d. Suitable equations for determining the heat flux are then

$$\left.\begin{aligned}
&u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y} = \frac{1}{R}\nabla^2\zeta, \quad R = \frac{U_\infty d}{\nu},\\
&u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{1}{\sigma R}\nabla^2 T, \quad \sigma = \frac{\rho\nu c_p}{k},\\
&\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} = \zeta,\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0,\\
&(u,v) = (0,0), \quad T = T_w \quad \text{for} \quad x^2+y^2 = \tfrac{1}{4},\\
&(u,v)\to(1,0), \quad T\to T_\infty \quad \text{for} \quad x^2+y^2\to\infty,
\end{aligned}\right\} \tag{2.1}$$

where (x,y) are cartesian co-ordinates, (u,v) is the velocity, ζ the vorticity and T the temperature.

The velocity is independent of temperature and has to be determined first. Kaplun (1957) and Proudman & Pearson (1957) have specified the velocity to the approximation we shall require. Their results are not presented, however, in a form convenient for immediate application to the temperature equation, and so will be derived again. Also a slightly different approach is adopted. Previously the region where x, y is $O(1)$ (the Stokes region) and the region where x, y is $O(R^{-1})$ (the Oseen region) were treated separately. Asymptotic expansions for the velocity were sought separately for $R \ll 1$ and given x, y and for $R \ll 1$ and given Rx, Ry and were linked by reordering the expansions in the zone where the regions of validity overlap. In this way were obtained two asymptotic expansions for the velocity in powers of $\log^{-1} R$, one valid for given x, y (the Stokes expansion) and the other at given Rx, Ry (the Oseen expansion). The alternative approach adopted here is to proceed from Oseen's equation by iteration. The velocity is derived directly from the appropriately modified Oseen equation, as in Lamb's solution. The no-slip condition is thereby assimilated without recourse to separate expansion near the cylinder and to matching processes, and the calculation is correspondingly somewhat shorter.

We begin, then, with Oseen's equation

$$\frac{\partial \zeta}{\partial x} = \frac{1}{R}\nabla^2\zeta \tag{2.2}$$

and represent the velocity by

$$u = 1+\chi+\frac{1}{R}\left(\frac{\partial \phi}{\partial x}-\frac{\partial \chi}{\partial x}\right), \quad v = \frac{1}{R}\left(\frac{\partial \phi}{\partial y}-\frac{\partial \chi}{\partial y}\right). \tag{2.3}$$

With a minor change, Lamb's approximate solution is

$$\chi = -2e^{\frac{1}{2}Rx}K_0(\rho)/D(\tfrac{1}{4}R), \quad \phi = \left(2\log\rho - \frac{R^2\cos\theta}{16\rho}\right)\Big/D(\tfrac{1}{4}R), \tag{2.4}$$

where ρ, θ are Oseen variables, with $\rho = R(x^2+y^2)^{\frac{1}{2}}/2$ and $\theta = 0$ in the downstream direction, and

$$D(\rho) = I_0(\rho)\,K_0(\rho)+I_1(\rho)\,K_1(\rho). \tag{2.5}$$

Here and below the I_n and K_n denote Bessel functions of imaginary argument defined as in Watson's well-known treatise. Since R is small,

$$D(\tfrac{1}{4}R) = -\log(\tfrac{1}{8}R)+\tfrac{1}{2}-\gamma+O(R^2\log R), \tag{2.6}$$

where $\gamma = 0{\cdot}577\ldots$ is Euler's constant. The approximation above differs from Lamb's only in the replacement of $-\log(\frac{1}{8}R)+\frac{1}{2}-\gamma$ by the combination $D(\frac{1}{4}R)$, which arises when the no-slip condition at the cylinder is satisfied precisely.

So far the velocity is given correct to $O(\log^{-1}R)$ uniformly over the region of flow, this being the highest accuracy Oseen's equation allows. For the purpose in hand, however, a uniform approximation is not required. In the Oseen region we want the velocity as accurately as is feasible, whereas in the Stokes region we can tolerate errors in velocity of $O(1)$. The reason is that the velocity is needed only to evaluate the convection terms, first in the vorticity equation and then in the temperature equation. In the Oseen region these convection terms will be evaluated to sundry orders of $\log^{-1}R$ times the accompanying diffusion terms, but in the Stokes region these convection terms are only of $O(R)$ times the accompanying diffusion terms and so may be neglected altogether. Consequently it becomes permissible to curtail Lamb's solution to

$$\chi = -2e^{\frac{1}{2}Rx}K_0(\rho)/D(\tfrac{1}{4}R), \quad \phi = 2\log\rho/D(\tfrac{1}{4}R). \tag{2.7}$$

The velocity so determined will be called the first Oseen approximation and is correct to $O(\log^{-1}R)$ in the Oseen region and correct to $O(1)$ in the Stokes region. (In the language of the matched expansion procedure, (2.7) is, in effect, the Oseen expansion for the velocity to $O(\log^{-1}R)$; and the Oseen expansion is adequate to cope with convection in the vorticity equation in both Oseen and Stokes regions because, after expansion, the convection terms drop out of the vorticity equation for the Stokes region.)

For the next velocity approximation, we rewrite the vorticity equation as

$$\frac{\partial \zeta}{\partial x}+\Delta = \frac{1}{R}\nabla^2\zeta, \tag{2.8}$$

where
$$\Delta = (u-1)\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}. \tag{2.9}$$

The added convection term Δ is to be evaluated from the first Oseen approximation and can be regarded here as known. The velocity is again to be represented by means of (2.3), and a fresh determination of ϕ and χ is to be made to incorporate Δ. After use of (2.3), the vorticity equation (2.8) yields

$$\left.\begin{aligned} \frac{\partial \chi}{\partial x} + \int_y^\infty \Delta\, dy &= \frac{1}{R}\nabla^2 \chi, \\ \int_y^\infty \Delta\, dy &= \frac{1}{R}\nabla^2 \phi, \end{aligned}\right\} \tag{2.10}$$

where Δ has been assumed integrable, and χ and ϕ have been chosen so that their derivatives vanish at infinity. We now put

$$\left.\begin{aligned} \int_y^\infty \Delta\, dy &= \tfrac{1}{4}R\, e^{\frac{1}{2}Rx} \sum_0^\infty \Delta_n(\rho) \cos n\theta \\ &= \tfrac{1}{4}R \sum_0^\infty \Delta_n^*(\rho) \cos n\theta. \end{aligned}\right. \tag{2.11}$$

The convection term Δ is thereby represented by Δ_n and Δ_n^*. When Δ is determined from the first Oseen approximation (2.7), we obtain

$$\Delta_n = 4D^{-2}\left\{\delta_{n0}\int_\infty^\rho \frac{K_1'(\rho)}{\rho}\, d\rho + \rho\delta_{n1}\int_\infty^\rho \frac{K_1(\rho)}{\rho^2}\, d\rho + (2-\delta_{n0})(\alpha I_n(\rho) + \beta\rho I_n'(\rho))\right\}, \tag{2.12}$$

where $2\alpha = K_0^2(\rho) - K_1^2(\rho)$, $\beta = \int_\infty^\rho \left[\frac{K_0(\rho)K_1(\rho)}{\rho} - K_1^2(\rho) - K_0(\rho)K_1'(\rho)\right]\frac{d\rho}{\rho}$

and δ_{ni} denotes Kronecker's delta. The Δ_n^* may then be determined from the Δ_n by

$$\Delta_n^* = \left(1 - \frac{\delta_{n0}}{2}\right)\sum_{m=0}^\infty \Delta_m[I_{n+m}(\rho) + I_{n-m}(\rho)]. \tag{2.13}$$

In the Oseen region, where $\rho = O(1)$, the Δ_n and Δ_n^* are all of $O(D^{-2}) = O(\log^{-2} R)$, corresponding to Δ being $O(\log^{-2} R)$ smaller than the diffusion term of the vorticity equation. Where ρ is $o(1)$,

$$\left.\begin{aligned} &\Delta_0 \text{ and } \Delta_0^* = O(\log^2\rho/\log^2 R), \\ &\Delta_n \text{ and } \Delta_n^* = O(\rho^{n-2}\log\rho/\log^2 R) \quad (n \geqslant 1). \end{aligned}\right\} \tag{2.14}$$

A suitable solution of (2.10) for ϕ and χ, and hence for u, v, is

$$\left.\begin{aligned} \chi &= e^{\frac{1}{2}Rx}\sum_0^\infty \chi_n(\rho)\cos n\theta, \\ \phi &= \sum_0^\infty \phi_n(\rho)\cos n\theta, \end{aligned}\right\} \tag{2.15}$$

where

$$\left.\begin{aligned}\chi_n &= AK_0(\rho)\,\delta_{n0}+I_n(\rho)\int_\infty^\rho \rho K_n(\rho)\,\Delta_n\,d\rho - K_n(\rho)\int_{\frac{1}{4}R}^\rho \rho I_n(\rho)\,\Delta_n\,d\rho \quad (n \geqslant 0),\\ \phi_0 &= B_0 \log\rho + \int_{\frac{1}{4}R}^\rho \rho^{-1}\left(\int_{\frac{1}{4}R}^\rho \rho\Delta_0^*\,d\rho\right)d\rho,\\ \phi_n &= B_1\rho^{-1}\delta_{n1} + \frac{\rho^n}{2n}\int_\infty^\rho \rho^{1-n}\Delta_n^*\,d\rho - \frac{\rho^{-n}}{2n}\int_{\frac{1}{4}R}^\rho \rho^{1+n}\Delta_n^*\,d\rho \quad (n \geqslant 1).\end{aligned}\right\} \tag{2.16}$$

The A, B_0 and B_1 are disposable constants, which will be fixed by satisfying the no-slip condition approximately. First, however, we consider separately the part of the solution that remains when these constants are zero, using a bar to distinguish the corresponding values of ϕ_n, χ_n, ϕ, χ, u and v. The particular solution $\bar{u}$, $\bar{v}$ so defined already accounts for the added convection term Δ in the vorticity equation and satisfies the boundary condition at infinity. In the Oseen region, $(\bar{u}, \bar{v})$ is $O(\log^{-2} R)$. For fixed ρ, which may be small,

$$\left.\begin{aligned}&\bar{\chi}_0 = O(1/\log^2 R), \quad \frac{d\bar{\chi}_0}{d\rho} \quad\text{and}\quad \frac{d\bar{\phi}_0}{d\rho} = O(\rho\log^2\rho/\log^2 R),\\ &\bar{\chi}_n, \quad \rho\frac{d\bar{\chi}_n}{d\rho}, \quad \bar{\phi}_n \quad\text{and}\quad \rho\frac{d\bar{\phi}_n}{d\rho} = O(\rho^n\log^2\rho/\log^2 R) \quad (n \geqslant 1).\end{aligned}\right\} \tag{2.17}$$

Thence, in the Stokes region, where $\rho = O(R)$, the particular solution $\bar{u}$, $\bar{v}$ is given by

$$\bar{u} = 1+\frac{1}{4}\int_0^\infty [\rho K_1(\rho)\,\Delta_1 - 2\rho K_0(\rho)\,\Delta_0 - \Delta_1^*]\,d\rho + O(R), \quad \bar{v} = O(R). \tag{2.18}$$

The integrand, incidentally, is bounded for small ρ, because the singularities in Δ_1 and Δ_1^* at $\rho = 0$ cancel. Thus, apart from terms of $O(R)$, the particular solution $\bar{u}$, $\bar{v}$ comprises simply the velocity at infinity plus a *uniform* streamwise velocity of $O(\log^{-2} R)$. To complete the solution, we need to subtract a solution of Oseen's equation (2.2) with zero velocity at infinity and the appropriate uniform velocity over the cylinder. This can be accomplished by using Lamb's solution, with a trivial rescaling to accommodate the integral of (2.18), and is given effect by putting

$$\left.\begin{aligned}A &= -2\left\{1+\frac{1}{4}\int_0^\infty [\rho K_1(\rho)\,\Delta_1 - 2\rho K_0(\rho)\,\Delta_0 - \Delta_1^*]d\rho\right\}/D(\tfrac{1}{4}R),\\ B_0 &= -A, \quad B_1 = R^2 A/32.\end{aligned}\right\} \tag{2.19}$$

The velocity determined by (2.15), (2.16) and (2.19) then vanishes at the cylinder to $O(R)$ and is uniformly correct to $O(\log^{-2} R)$, which is as accurate as one iteration of Oseen's equation allows.

It is not proposed to consider higher approximations. Further, as mentioned before, errors of $O(1)$ in the velocity can be tolerated in the Stokes region for the present application. So, for simplicity, we shall take $B_1 = 0$. With this modification, the velocity just derived is accurate to $O(\log^{-2} R)$ in the Oseen region and is accurate to $O(1)$ in the Stokes region. We shall refer to this modified approximation as the second Oseen approximation.

Whilst the emphasis here is on the specific application to hot wires, the derivation of a higher-order approximation to velocity has an underlying theme which is strikingly simple, and, though no longer new, it should not perhaps be passed over without comment. The salient points emerging here and brought out earlier, particularly clearly by Kaplun (1957), are that (i) in an approximation in which velocities of $O(\log^{-n} R)$ are retained and velocities of $O(R)$ are neglected, vorticity convection is significant only in the Oseen region; (ii) when convection terms (presumed known from previous approximation to velocity) are inserted in Oseen's equation, they can be accommodated by a particular solution $(\overline{u}, \overline{v})$ for velocity which satisfies the boundary condition at infinity and which, because of the relatively large region in which convection is effective, is approximately *uniform* ($= \mathbf{u}_0$ say) over the entire Stokes region. The complete velocity approximation appropriate to the convection terms inserted in Oseen's equation is then, of course, given by adding to $(\overline{u}, \overline{v})$ Lamb's solution of Oseen's equation for a cylinder moving with velocity $-\mathbf{u}_0$ in still fluid.

3. Temperature approximation

Similar approximations are now evolved for the heat equation. We write

$$\frac{\partial T}{\partial x} + \Delta_T = \frac{1}{\sigma R} \nabla^2 T, \tag{3.1}$$

where

$$\Delta_T = (u-1)\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{R}{4} e^{\frac{1}{2}\sigma x R} \sum_{n=0}^{\infty} \Delta_{Tn} \cos n\theta. \tag{3.2}$$

If at a stage of iteration T is expressed as

$$T = T_\infty + e^{\frac{1}{2}\sigma x R} \sum_{n=0}^{\infty} T_n(\rho) \cos n\theta, \tag{3.3}$$

and ϕ and χ are as in (2.15), then the coefficients in (3.2) are defined by

$$2\Delta_{Tn} = \Phi_n + (1 - \tfrac{1}{2}\delta_{n0}) \sum_{m=0}^{\infty} X_m [I_{m-n}(\rho) + I_{m+n}(\rho)],$$

where

$$\left.\begin{aligned} X_n = {} & (1-\tfrac{1}{2}\delta_{n0})[2\sigma(\chi_0 T_n + T_0 \chi_n) - 2\chi_0' T_n' - 2T_0' \chi_n' - \sigma\chi_0'(T_{n-1} + T_{n+1}) \\ & + T_0'(\chi_{n-1} + \chi_{n+1}) + \chi_0[T_{n-1}' + T_{n+1}' + (1/\rho)\{(1-n)T_{n-1} + (1+n)T_{n+1}\}] \\ & - \sigma T_0[\chi_{n-1}' + \chi_{n+1}' + (1/\rho)\{(1-n)\chi_{n-1} + (1+n)\chi_{n+1}\}]], \\ \Phi_n = {} & (1-\tfrac{1}{2}\delta_{n0})[2(\phi_0' T_n' + \phi_n' T_0') + \sigma\phi_0'(T_{n-1} + T_{n+1}) \\ & + \sigma T_0[\phi_{n-1}' + \phi_{n+1}' + (1/\rho)\{(1-n)\phi_{n-1} + (1+n)\phi_{n+1}\}]], \end{aligned}\right\} \tag{3.4}$$

with

$$\phi_{-1} \equiv \phi_1, \quad \chi_{-1} \equiv \chi_1, \quad T_{-1} \equiv T_1.$$

Products $\phi_n T_m$ and $\chi_n T_m$ with both n and m greater than zero are omitted because only the first three iterative approximations to the heat transfer are to be calculated. For this purpose only the two leading non-trivial iterative approximations to Δ_T and hence to Δ_{Tn} are needed. The leading approximations to u, v and T involve ϕ_0, χ_0 and T_0, cf. (2.7) and (3.7), and the higher-order ϕ_n, χ_n and T_n ($n \geqslant 1$) respectively are at least one order of $\log^{-1} R$ smaller. Thus the products referred to are at least two orders smaller than the corresponding leading products $\phi_0 T_0$ and $\chi_0 T_0$ and can be neglected. The same is equally true of the products $\phi_n T'_m$, etc., with n and m greater than zero, which include derivatives.

The approximation succeeding (3.3) can now be written as

$$T = T_\infty + e^{\frac{1}{2}\sigma x R} \sum_{n=0}^{\infty} \left[\alpha_n K_n(\sigma\rho) + \sigma I_n(\sigma\rho) \int_\infty^\rho \rho\Delta_{Tn} K_n(\sigma\rho)\, d\rho - \sigma K_n(\sigma\rho) \int_{\frac{1}{4}R}^\rho \rho \Delta_{Tn} I_n(\sigma\rho)\, d\rho\right] \cos n\theta, \tag{3.5}$$

where, to make $T = T_w$ at the cylinder, we take

$$\alpha_n = \frac{I_n(\frac{1}{4}\sigma R)}{K_n(\frac{1}{4}\sigma R)} \left\{(-2)^n (T_w - T_\infty) + \sigma \int_{\frac{1}{4}R}^\infty \rho \Delta_{Tn} K_n(\sigma\rho)\, d\rho\right\} \quad (n \geqslant 0) \tag{3.6}$$
$$\simeq 0 \quad \text{for} \quad n \geqslant 1.$$

The error in temperature due to assuming α_n to be zero for $n \geqslant 1$ is $O(R)$ and will be neglected.

The first three approximations to T can now be enumerated. The 'Oseen approximation' corresponding to $\Delta_T = 0$ will be taken as

$$T^{(1)} = T_\infty + \alpha_0^{(1)} e^{\frac{1}{2}\sigma x R} K_0(\sigma\rho), \tag{3.7}$$

with

$$\alpha_0^{(1)} = (T_w - T_\infty) \frac{I_0(\frac{1}{4}\sigma R)}{K_0(\frac{1}{4}\sigma R)}, \tag{3.8}$$

which is a result given first in a different form but with an error of the same order in R by Cole & Roshko (1954). For the second approximation, Δ_T $(=\Delta_T^{(1)}$ say) will be evaluated from (3.4) with u, v and T defined by the first Oseen approximations, (2.7) and (3.7). Accordingly the components $\Delta_{Tn}^{(1)}$ associated with $\Delta_T^{(1)}$ become

$$\Delta_{Tn}^{(1)} = 2\sigma A\, \alpha_0^{(1)} \{K_0(\sigma\rho) \{(1 - \tfrac{1}{2}\delta_{n0}) [I_n(\rho) K_0(\rho) + I'_n(\rho) K_1(\rho)] - \tfrac{1}{2}\delta_{n1}/\rho\} - K_1(\sigma\rho)(1 - \delta_{n0}) [I_n(\rho) K_1(\rho) + I'_n(\rho) K_0(\rho)]\}. \tag{3.9}$$

The second approximation to T and α (denoted by $T^{(2)}$ and $\alpha^{(2)}$) is now given by (3.5) and (3.6) with the Δ_{Tn} represented by (3.9). For the third approximation, Δ_T $(=\Delta_T^{(2)})$ is evaluated from the second Oseen approximation to the velocity specified in the preceding section and from the $T^{(2)}$ just defined (or more precisely from the concomitant $T_n^{(2)}$). After some algebra, (3.4) then yields for the components $\Delta_{Tn}^{(2)}$ of $\Delta_T^{(2)}$:

$$\left.\begin{aligned}
\Delta_{T0}^{(2)} &= \sigma\alpha_0^{(2)} \left[A\, K_0(\sigma\rho)(I_0 K_0 + I_1 K_1) - \frac{4K_1(\sigma\rho)}{\rho} \int_0^\rho \rho \Delta_0^*\, d\rho + 2K_0(\sigma\rho) \int_\infty^\rho \Delta_1^*\, d\rho\right] \\
&\quad + \frac{\sigma^2 A}{2\rho} [K_1(\sigma\rho)\, i_{T1} - I_1(\sigma\rho)\, k_{T1}] + \sum_{n=0}^{\infty} (a_n k_{Tn} + b_n i_{Tn} + c_n k_n + d_n i_n) \\
&= O[A\, \alpha_0^{(1)} \log^2 \rho (1 + A \log \rho)] \\
&= O[\log^2 \rho / \log^2 R], \\
\Delta_{Tn}^{(2)} &= O[A\, \alpha_0^{(1)} \rho^{n-2} \log \rho (1 + A \log \rho)] \quad (n \geqslant 1) \\
&= O[\rho^{n-2} \log \rho / \log^2 R],
\end{aligned}\right\} \tag{3.10}$$

where

$$\left.\begin{aligned} b_n &= \sigma^2 A[K_0\{\tfrac{1}{2}(1-\delta_{n0})[I_{n-1}K_{n-1}(\sigma\rho)+I_{n+1}K_{n+1}(\sigma\rho)]-I_nK_n(\sigma\rho)\} \\ &\quad -K_1\{(1-\delta_{n0})I_nK_n'(\sigma\rho)+I_n'K_n(\sigma\rho)\}], \\ d_n &= \sigma\alpha_0^{(2)}\left[\frac{K_1(\sigma\rho)}{\rho}-K_0(\sigma\rho)\{\tfrac{1}{2}(I_{n-1}K_{n-1}+I_{n+1}K_{n+1})+I_nK_n\}\right] \end{aligned}\right\} \quad (3.11)$$

and

$$\left.\begin{aligned} k_{Tn} &= \int_\infty^\rho \rho\Delta_{Tn}^{(1)}K_n(\sigma\rho)\,d\rho, & i_{Tn} &= \int_0^\rho \rho\Delta_{Tn}^{(1)}I_n(\sigma\rho)\,d\rho, \\ k_n &= \int_\infty^\rho \rho\Delta_n K_n\,d\rho, & i_n &= \int_0^\rho \rho\Delta_n I_n\,d\rho, \end{aligned}\right\} \quad (3.12)$$

the I_r and K_r having arguments ρ unless otherwise indicated.

The expression for c_n is the same as for d_n with the $K_1(\sigma\rho)/\rho$ omitted and the $K_{n+r}(\rho)$, $r = -1, 0, 1$ replaced by $(-1)^{r+1}I_{n+r}(\rho)$. Likewise the expression for a_n is the same as for b_n but with $(-1)^{r+1}I_{n+r}(\sigma\rho)$ instead of $K_{n+r}(\sigma\rho)$.

The Δ_n were defined in (2.12) and Δ_0^* and Δ_1^* by (2.13). The order estimates for the $\Delta_{Tn}^{(2)}$ apply also to the preceding approximation $\Delta_{Tn}^{(1)}$. The $\Delta_{Tn}^{(2)}$ with $n \geqslant 1$ are not specified explicitly because the numerical computation is confined to the heat flux from the cylinder which is seen immediately below to involve only $\Delta_{T0}^{(2)}$.

Turning now to the heat transfer from the cylinder, we have

$$\left.\begin{aligned} N &= \frac{R}{8\pi(T_\infty-T_w)}\int_0^{2\pi}\frac{\partial T}{\partial\rho}\bigg|_{\rho=\frac{1}{4}R}d\theta \\ &= \frac{\sigma R}{8(T_\infty-T_w)}\bigg\{\alpha_0[K_0(\tfrac{1}{4}\sigma R)\,I_1(\tfrac{1}{4}\sigma R)-K_1(\tfrac{1}{4}\sigma R)\,I_0(\tfrac{1}{4}\sigma R)] \\ &\quad +2\sigma\sum_{n=0}^{\infty}I_n(\tfrac{1}{4}\sigma R)\,I_n'(\tfrac{1}{4}\sigma R)\int_\infty^{\frac{1}{4}R}\rho\Delta_{Tn}K_n(\sigma\rho)\,d\rho\bigg\} \\ &= \alpha_0/(T_w-T_\infty)+O(R^2\log R) \\ &= \frac{I_0(\frac{1}{4}\sigma R)}{K_0(\frac{1}{4}\sigma R)}\left[1+\frac{\sigma}{T_w-T_\infty}\int_0^\infty \rho\Delta_{T0}K_0(\sigma\rho)\,d\rho\right]+O(R^2\log R), \end{aligned}\right\} \quad (3.13)$$

where use has been made of the orders of magnitude (3.10) of the relevant approximations to Δ_{Tn}. Thence, after successively substituting 0, $\Delta_{T0}^{(1)}$ and $\Delta_{T0}^{(2)}$ for Δ_{T0} and after some computation, we get the successive approximations to Nusselt number:

$$\left.\begin{aligned} N_1 &= I_0(\tfrac{1}{4}\sigma R)/K_0(\tfrac{1}{4}\sigma R) \quad (=Q,\ \text{say}), \\ N_2 &= N_1-\lambda Q^2/D(\tfrac{1}{4}R), \\ N_3 &= N_2-\mu Q^2/D^2(\tfrac{1}{4}R), \end{aligned}\right\} \quad (3.14)$$

where

$$\lambda = 2\sigma^2\int_0^\infty \rho K_0^2(\sigma\rho)\,D(\rho)\,d\rho,$$

the error in each N_r being $O(\log^{-r-2}R)$. For a Prandtl number $\sigma = 0{\cdot}72$ the numerical coefficients λ and μ were found to be $\lambda = 1{\cdot}38$ and $\mu = 0{\cdot}40$. The variation in λ over a small range of neighbouring values of σ is shown in table 1. The second coefficient '0·40' yields only 6 % of Nu for $R = 0{\cdot}4$ and for smaller R

mostly yields much less. For this reason and the length of computation involved, the sensitivity of μ to variations in σ was not investigated.

Fortunately, the contributions to the second numerical coefficient from the terms $a_n k_{Tn} + b_n i_{Tn} + c_n k_n + d_n i_n$ in the expression for $\Delta_{T0}^{(2)}$ diminished extremely rapidly with increasing n. For $n \geqslant 1$ these contributions, multiplied by a factor 4, were

$(a_1 k_{T1})$	1·77	$(b_1 i_{T1})$	−0·14	$(c_1 k_1)$	−0·51	$(d_1 i_1)$	−0·46
$(a_2 k_{T2})$	0·01	$(b_2 i_{T2})$	−0·001	$(c_2 k_2)$	−0·01	$(d_2 i_2)$	−0·001
$(a_3 k_{T3})$	0·0001						

where the source is exhibited in brackets. Preliminary estimates of the higher-order $a_n k_{Tn}$, $b_n i_{Tn}$, $c_n k_n$ and $d_n i_n$ indicated that the further contributions were insignificant, and only those displayed were computed at all accurately.

σ	λ
0·6	1·250
0·7	1·363
0·8	1·466
0·9	1·560
1·0	1·644

TABLE 1. Coefficient λ required for N_2

The second and third approximations are plotted in figure 1 together with a smoothed curve from the measured data of Collis & Williams (1959) in which allowance has been made for three-dimensional effects and for temperature jump at the cylinder. The agreement at Reynolds numbers less than 0·3 is within about 1 % in N. Precise error bounds are not available, either for the calculation or the experiments, but it seems probable that the second and third approximations are, at least for $R < 0{\cdot}3$, a substantial improvement on the simple 'first Oseen' approximation N_1, which was derived in a reduced form of equivalent accuracy by Cole & Roshko (1954) and was the best previous result.

All these approximations have been derived on the understanding that both R and σR are small, so, for fluids with large Prandtl number, the maximum Reynolds numbers for which the approximations hold can be expected to be correspondingly reduced.

Finally, attention is drawn to the composition of the approximations for temperature (which closely parallels that of the higher approximations for velocity). The main elements in their derivation are that: (i) to an approximation in which temperatures of $O(\log^{-n} R)$ are retained and temperatures of $O(R)$ are neglected, thermal convection is significant only in the Oseen region; (ii) when corrective convection terms Δ_T (presumed known from previous approximation to temperature) are inserted in the thermal Oseen equation, they can be accommodated by a particular solution for temperature $\overline{T}$ which satisfies the condition at infinity and which, because again of the relatively large extent of the region in which thermal convection is effective, is approximately *uniform* ($= t_0$ say) over the entire Stokes region; (iii) the temperature approximation defined from Oseen's equation with Δ_T presumed known may be adequately completed by adding to the particular solution $\overline{T}$ an approximate solution of the thermal Oseen equation, say $T = -t_0 N_1 e^{\frac{1}{2}\sigma x R} K_0(\sigma\rho)$, which gives $T = -t_0 + O(R)$ at the cylinder and $T = 0$ at infinity.

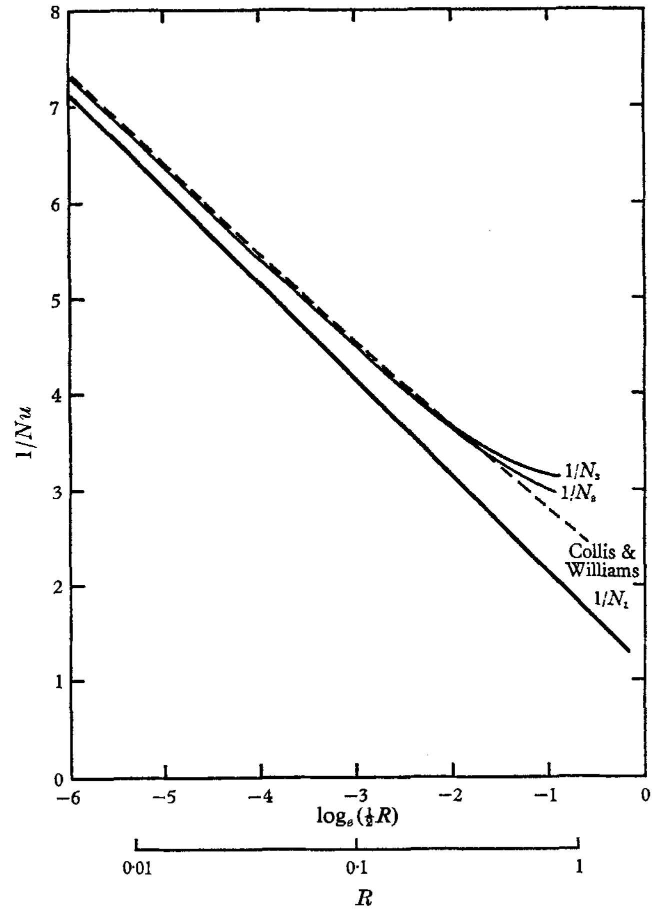

FIGURE 1. Heat transfer approximations.

The particular solution $\overline{T}$ just referred to can be taken to be given by (3.5) with all the α_n zero. Then, in the Stokes region

$$\overline{T} = T_\infty - \sigma \int_0^\infty \rho \Delta_{T0} K_0(\sigma\rho)\, d\rho + O(R), \tag{3.15}$$

which apart from terms of $O(R)$ is uniform, as noted.

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